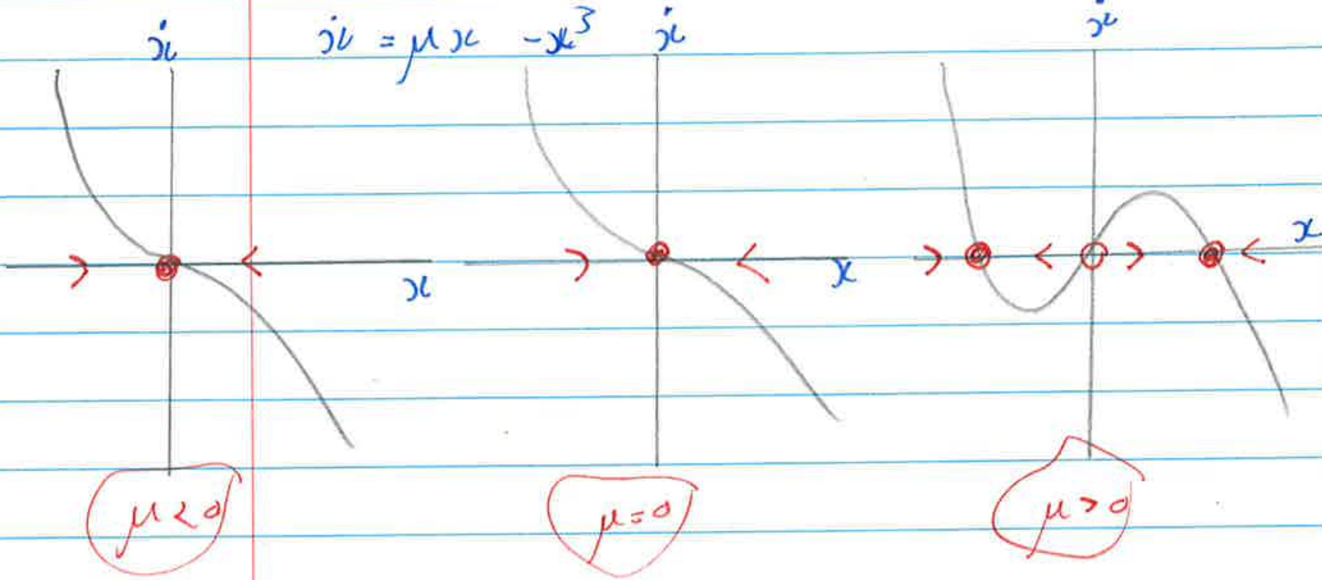
Summary of Hopf Bifurcation

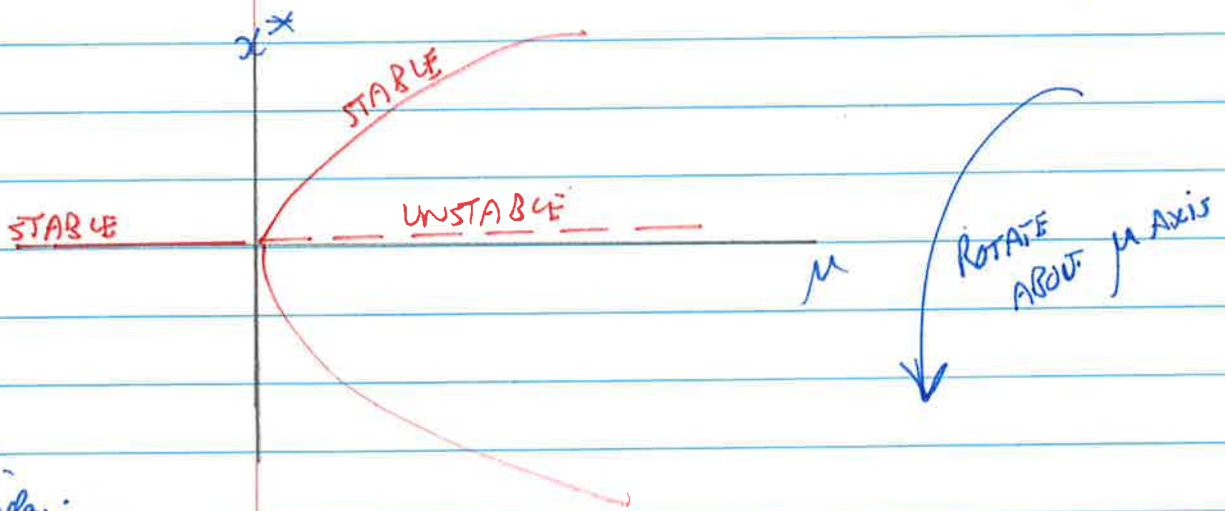
8/12/25

Recall 1d Pitchfork bifurcation:

$\dot{x} = \mu x - x^3$



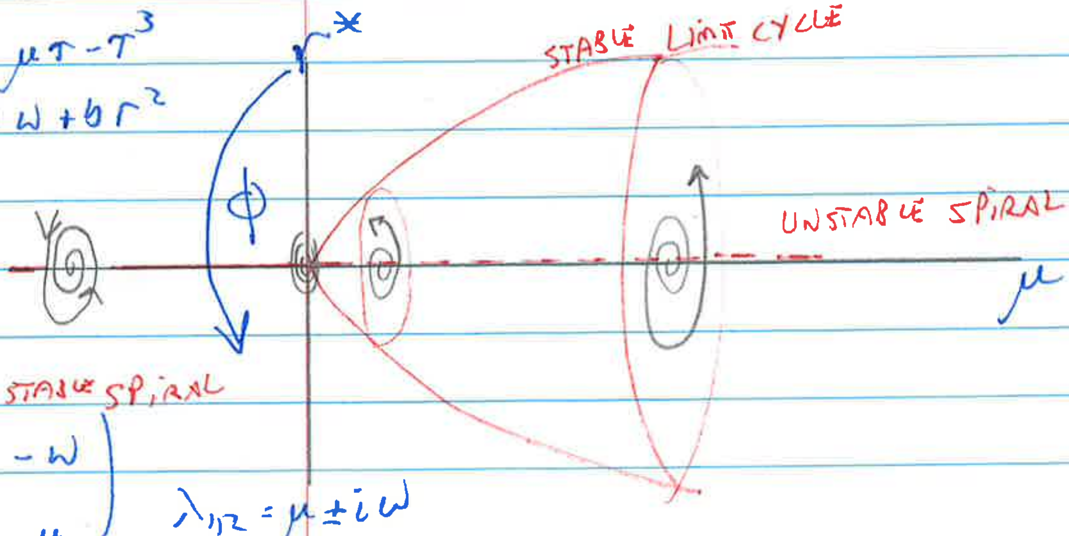
Q. what is the difference between FP at $x=0$ for $\mu<0$, $\mu=0$?



Now consider:

$\dot{r} = \mu r - r^3$

$\dot{\phi} = \omega + b r^2$



$$J = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

$\lambda_{1,2} = \mu \pm i\omega$

Comparison of Dynamics in 1d, 2d, 3d

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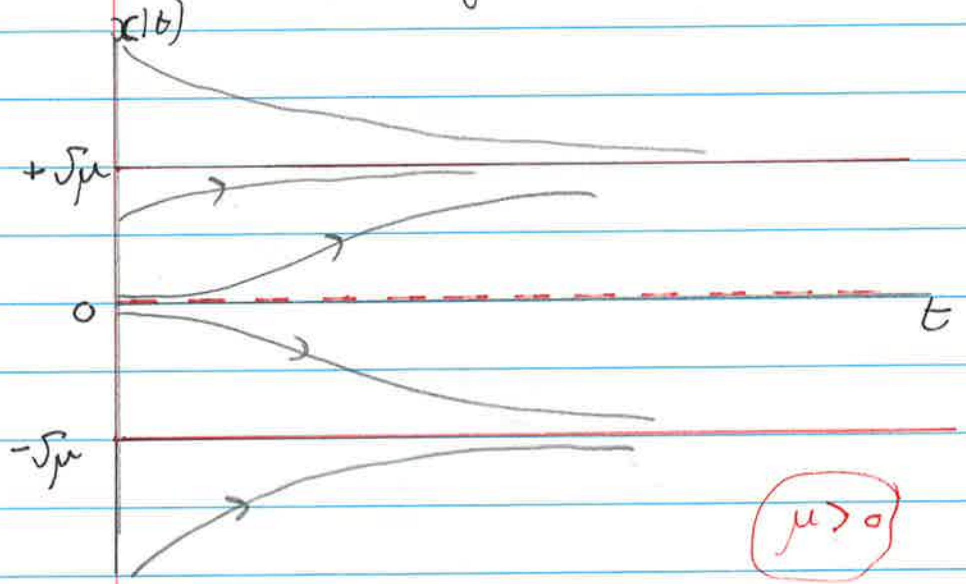
$\dot{x} = \mu x - x^3$



Q. How is life so robust?

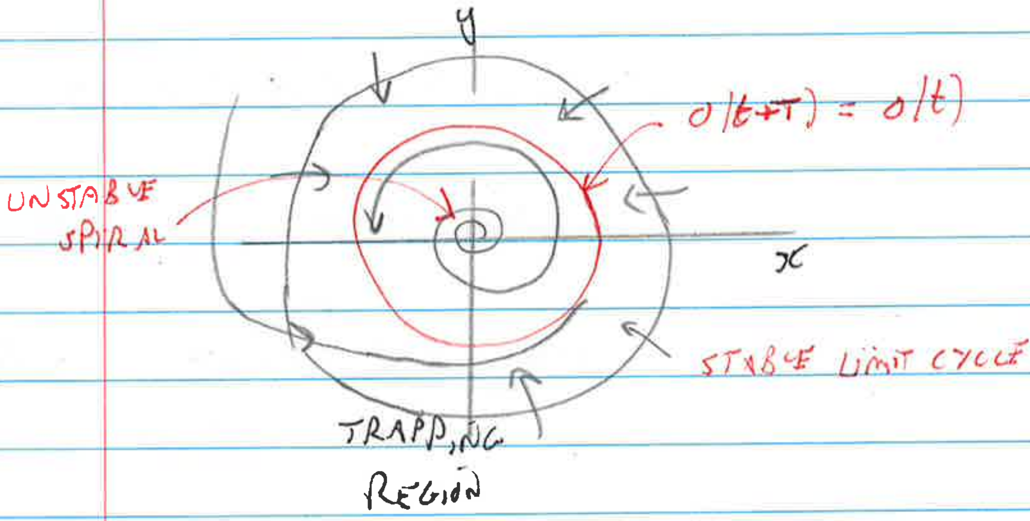
1d Trajectories are monotonic, FPs stop motion

2d

e.g. Glycolysis cycle

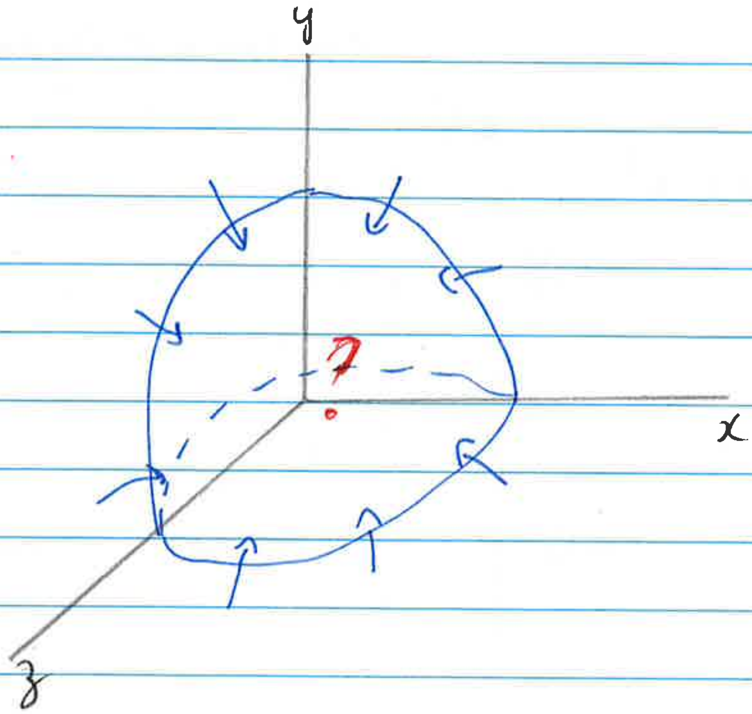
$\dot{x} = -x + ay + x^2 y$

$\dot{y} = b - ay - x^2 y$



Trajectories can now oscillate but still confined to a region must approach a FP or limit cycle - they cannot escape being trapped between other trajectories.

3d



Suppose we have: - a Trapping region (~sphere)
- No stable node or spiral inside
- No limit cycle

Q Where do the trajectories go? STRANGE ATTRACTOR

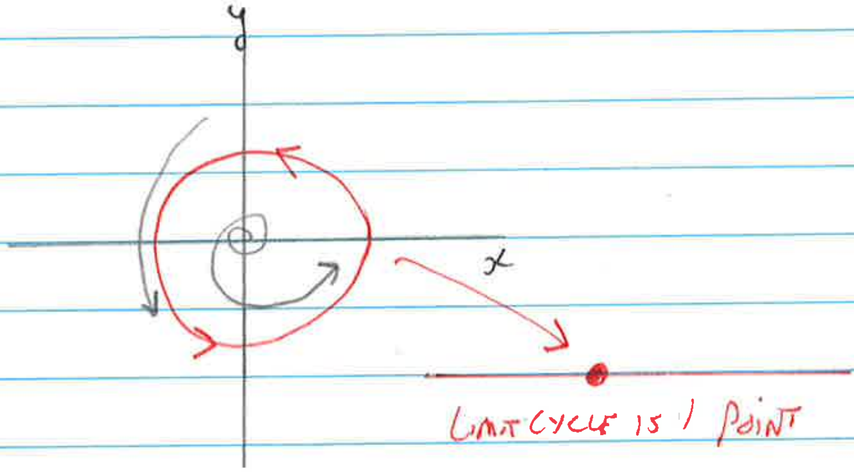
Whatever chaos is, it is not periodic

Lorenz 1963 discovered Deterministic Non-Periodic Flow

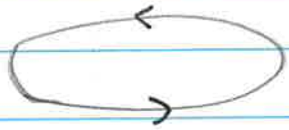
But 3D dynamical systems cannot be treated using the methods of 2d.

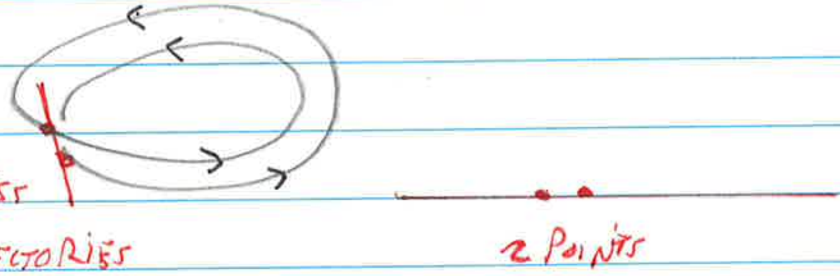
Poincaré map

2d

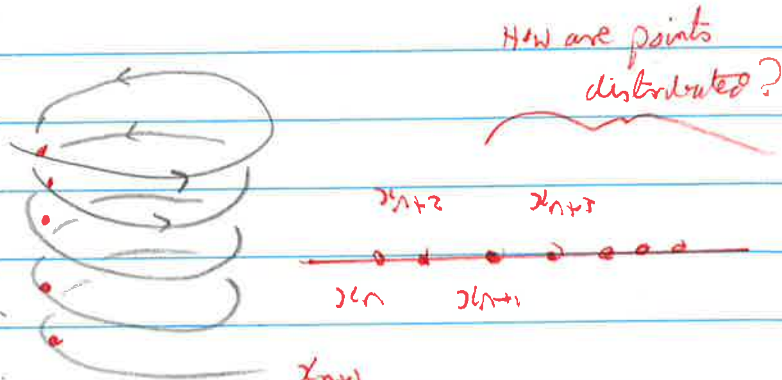


3d

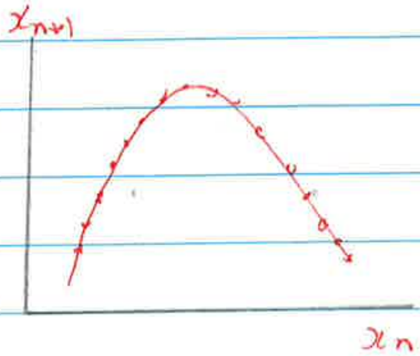




Near miss of trajectories



Lot of near misses



Period doubling

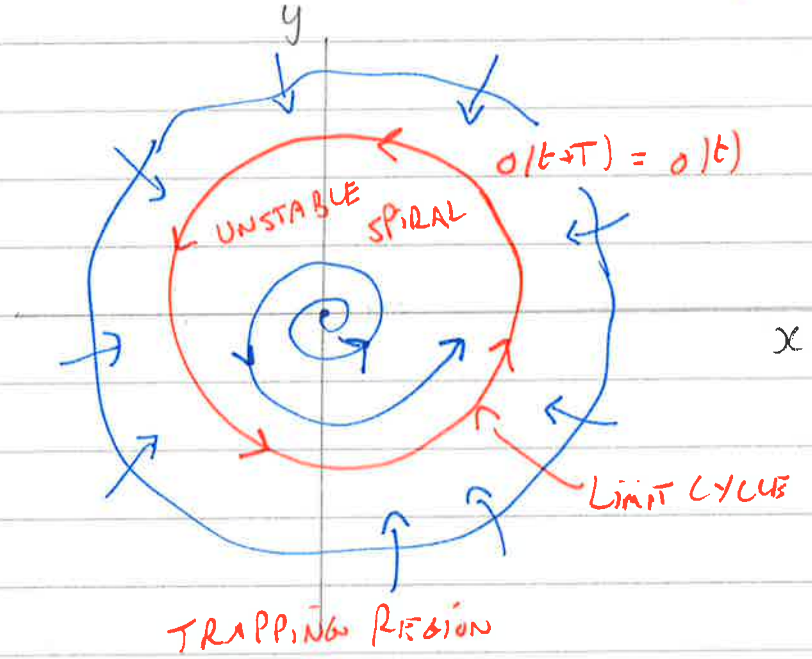
We study it using 1d discrete iterated maps

Lecture 13 CHAOS

3/12/24 What is chaos? Whatever it is, it is NOT periodic.

2D/ $\dot{x} = f(x, y)$

$\dot{y} = g(x, y)$

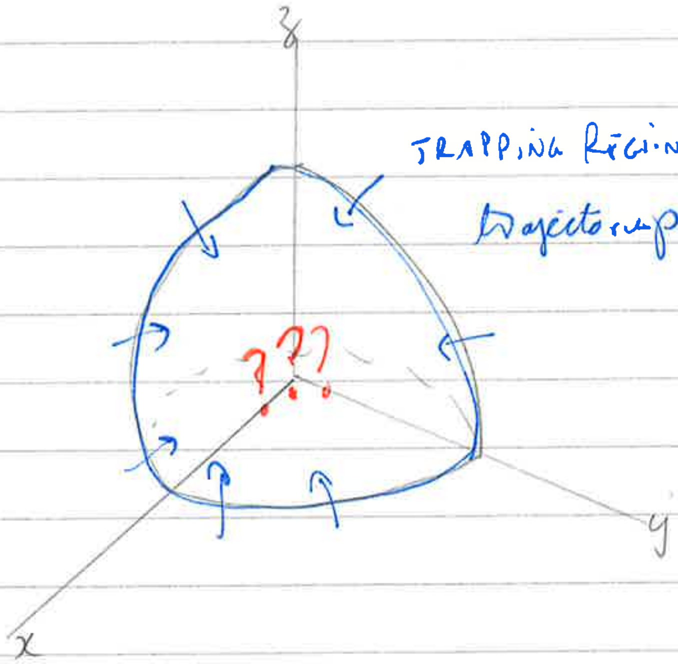


POINCARE – BENDIXSON THM.

3D/ $\dot{x} = f(x, y, z)$
$\dot{y} = g(x, y, z)$
$\dot{z} = h(x, y, z)$

But suppose inside there is no:

- stable node or spiral
- Limit cycle



TRAPPING REGION : ALL trajectories point inwards

Trajectories cannot intersect ⇒ A new type of attractor – STRANGE ATTRACTOR only exists in 3D or higher.

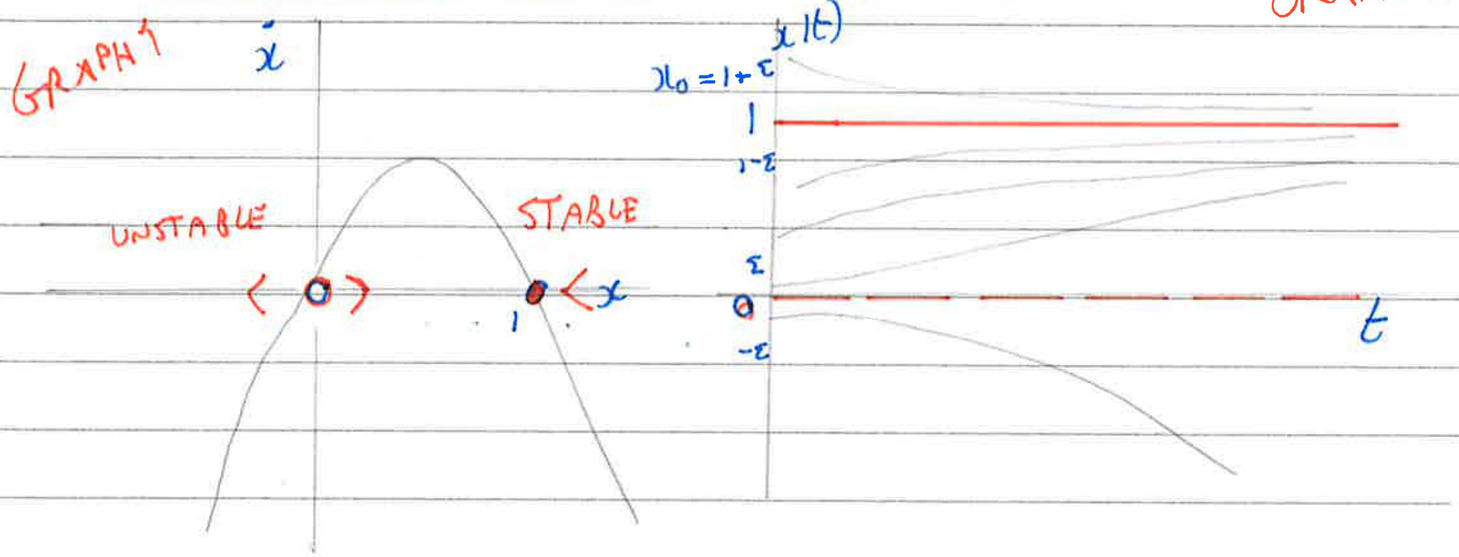
see Lorenz 1963 Deterministic non-Periodic Flow

3D is too complicated, so we look at 1D iterated maps.

So far we have looked at continuous dynamical systems

$$\boxed{\dot{x} = r\,x\,(1-x)}$$

Logistic Equation.



We can find trajectories without solving the equation

What is the initial slope of $x(t)$? obviously it is $\dot{x}(t=0)$

Let $x_0 = \varepsilon \sim 0 \Rightarrow \dot{x}(\varepsilon) = r\varepsilon(1-\varepsilon) \sim r\varepsilon \to 0$ as $\varepsilon \to 0$ i.e. Flat

NB If $x_0 = -\varepsilon$ $\quad x(t) \to -\infty$

Let $x_0 = 1-\varepsilon \Rightarrow \dot{x}(\varepsilon) = r(1-\varepsilon)(1-(1-\varepsilon)) \approx r\varepsilon \to 0$ as $\varepsilon \to 0$ i.e. Flat

Because the slope is flat at $x_0 = 0$ and $x_0 = 1$, there is a maximum in between, at $x = 1/2$.

Let $x_0 = 1+a$ (a not necessarily small), $\Rightarrow \dot{x} = r(1+a)(1-(1+a)) \sim -r.a < 0$

This is a simple equation, we know almost everything about its solutions.

Q. Do all "simple" equations have simple solutions?

DISCRETE ITERATED MAPS

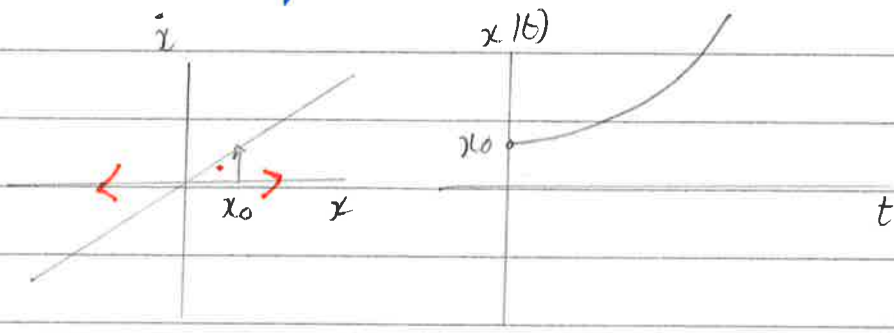
≡ Discrete equivalent of a function $f(x)$ in a 1D continuous dynamical system:

$$\dot{x} = f(x) \quad \rightarrow \quad x_{n+1} = f(x_n) \quad , \text{ given } x_0$$

Iteration replaces time in the map.

EXAMPLE 1

$$\boxed{\dot{x} = \alpha x \, , \ \alpha \neq 0}$$



∴ Always unstable for $\alpha \neq 0$,

Discrete case: $\boxed{x_{n+1} = \alpha x_n}$ NB This is linear

Given x_0

$$x_1 = \alpha x_0$$
$$x_2 = \alpha x_1 = \alpha^2 x_0$$
$$x_3 = \alpha x_2 = \alpha^3 x_0$$
$$\vdots$$
$$x_n = \alpha^n x_0$$

And if $|\alpha| < 1$, $\lim_{n\to\infty} x_n = 0$ ∴ It is stable

The discrete linear map is stable for $|\alpha| < 1$, unlike the ODE.*

$x^* = 0$ is the only fixed point.

*which is stable if $f'(x^*) < 0$.

EXAMPLE 2 $x_{n+1} = 1 - \alpha x_n$ NB This is still linear

$x_1 = 1 - \alpha x_0$

$x_2 = 1 - \alpha x_1 = 1 - \alpha(1 - \alpha x_0) = 1 - \alpha + \alpha^2 x_0$

$x_3 = 1 - \alpha x_2 = 1 - \alpha(1 - \alpha + \alpha^2 x_0) = 1 - \alpha + \alpha^2 - \alpha^3 x_0$

$x_4 = 1 - \alpha x_3 = 1 - \alpha(1 - \alpha + \alpha^2 - \alpha^3 x_0) = 1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 x_0$

$$\boxed{\therefore x_n = \left[\sum_{j=0}^{n-1} (-\alpha)^j\right] + (-\alpha)^n x_0}$$

But $\sum_{j=0}^{n-1} x^j = 1 + x + x^2 + \dots x^{n-1} = \dfrac{1 - x^n}{1 - x}$

$$\Rightarrow x_n = \frac{1 - (-\alpha)^n}{1 - (-\alpha)} + (-\alpha)^n x_0 = \frac{1 - (-\alpha)^n}{1 + \alpha} + (-\alpha)^n x_0$$

and if $|\alpha| < 1$, the $(-\alpha)^n$ terms go to zero:

$$\boxed{\therefore x^* = \lim_{n \to \infty} x_n = \frac{1}{1 + \alpha}}$$

Fixed point is independent of x_0!

Even this linear case requires ~1 page of algebra to solve.

Definition: A fixed point x^* of a discrete iterated map $x_{n+1} = f(x_n, r)$ is:

$$\boxed{x^* = f(x^*, r)}$$

i.e. You put x^* in and you get x^* out

Stability of a fixed point x^*:

Given $x_{n+1} = f(x_n, r)$, and $x^* = f(x^*, r)$

Let $x_n = x^* + \delta_n$
$x_{n+1} = x^* + \delta_{n+1}$

we want to see if the $\delta_n \to 0$ as $n \to \infty$?

$$\Rightarrow \cancel{x^*} + \delta_{n+1} = f(x^* + \delta_n, r) \sim \cancel{f(x^*, r)} + f'(x^*, r).\delta_n + \cdots$$

$$\boxed{\therefore \delta_{n+1} = \delta_n \cdot f'(x^*, r)}$$

NB Even though the map is discrete, the function $f(x, r)$ is continuous and differentiable.

Let $\lambda = |f'(x^*, r)|$

$\Rightarrow \delta_{n+1} = \lambda \delta_n \to \lambda^{n+1} \delta_0$, $\delta_0 = x_0 - x^*$

$\therefore$ If $\lambda < 1$, $\delta_{n+1} \to 0$ and x^* is a stable fixed point.

Condition for stability of x^*: $\left| \left. \frac{df}{dx} \right|_{x^*} \right| < 1$

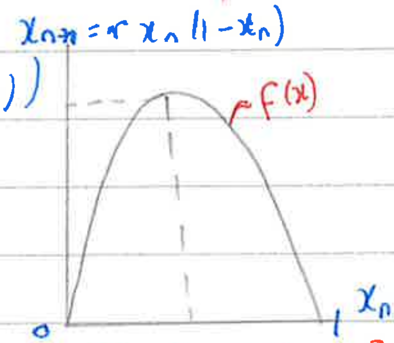
EXAMPLE 3: Discrete Logistic Map (simplest non-linear case)

$$x_{n+1} = r x_n (1 - x_n) \equiv f(x_n, r) \qquad 0 \leq r \leq 4$$

$$0 \leq x_n \leq 1$$

Ensures x_n stays in $(0,1)$

given $x_0 \in (0,1)$.



$$x_1 = r x_0 (1 - x_0)$$

$$x_2 = r x_1 (1 - x_1) = r^2 x_0 (1 - x_0)(1 - r x_0 (1 - x_0))$$

$$= r^2 x_0 (1 - x_0)(1 - r x_0 + r x_0^2)$$

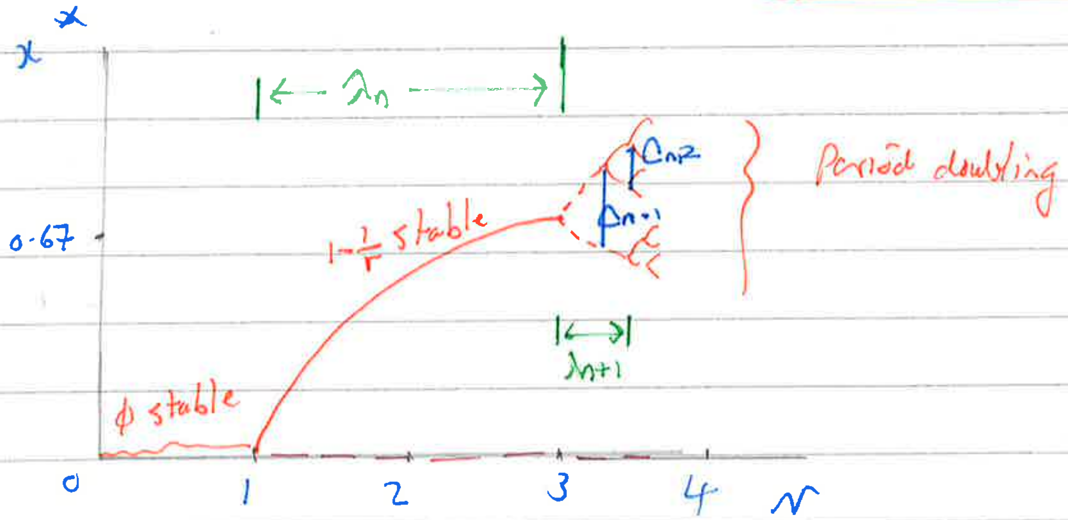
$$x_3 = \cdots ?$$

Repeat this N times; what fixed points do you get?

we need $x^* = f(x^*, r)$

$$\Rightarrow x^* = r x^* (1 - x^*) \qquad \boxed{\therefore x^* = 0} \text{ or } 1 = r(1 - x^*)$$

$$\boxed{\therefore x^* = 1 - 1/r} \text{ only if } r > 1$$



Stability?

$$f'(x^*) = r(1 - 2x^*) \Rightarrow f'(0) = r \ \therefore \text{stable if } r < 1$$

$$f'(1 - \tfrac{1}{r}) = (2 - r) \ \therefore \text{stable for } 1 < r < 3$$

We see that as r increases, first the origin becomes unstable, then the other fixed point of $f(x_n, r)$.
When $r > 3$, look at $f(f(x_n, r))$, what are its fixed points?

$$x^* = f(f(x^*, r)) = r^2 x^*(1-x^*)(1 - rx^* + rx^{*2})$$

$x^* = 0$ is still a fixed point, but we need a computer for others!

Note the slope of fixed point of f_n becomes unstable as r increases and new fixed points appear.

Feigenbaum Numbers

$$\delta_n = \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+2} - \lambda_{n+1}} \sim 4.6692\ldots$$ change in r parameter to next splitting.

$$\alpha_n = \frac{\Delta_{n}}{\Delta_{n+1}} \sim 2.503\ldots$$ splitting of the fixed points.

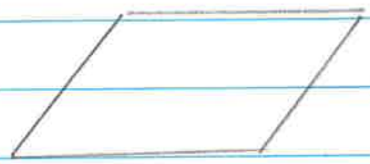
Universality in chaos

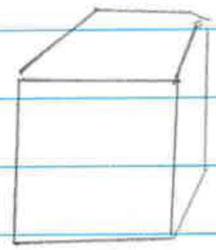
The fixed points of 1D discrete iterated maps not only don't depend on x_0, they don't depend on the function $f(x_n, r)$? They do depend on some properties but not details: Exercise 13 examines the Sine maps, which has the same behaviour as logistic map.

Chaos: What do we mean by DIMENSION? 6

Simple View:

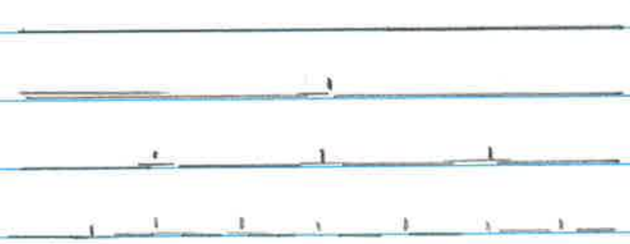
1d LINE

2d  PLANE

3d  VOLUME

Generalise this to: How does the size of an object scale with the size of the ruler we use to measure it?

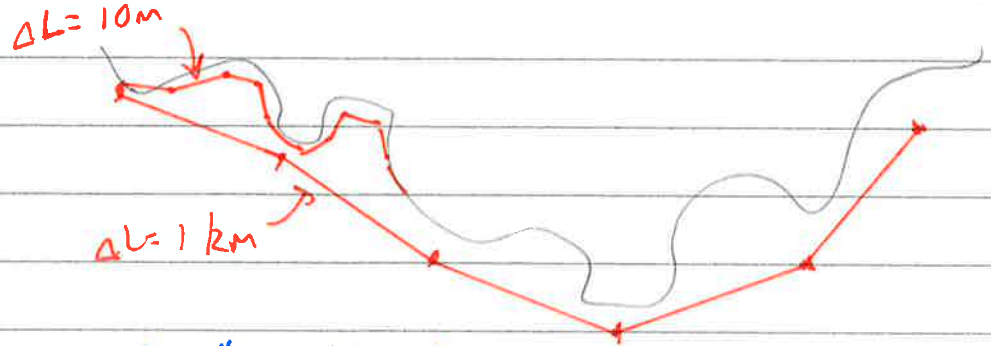
or What is the length of a coastline?



obviously $N(\varepsilon) \sim \frac{1}{\varepsilon}$

Assume: $N(\varepsilon) \sim \frac{1}{\varepsilon^D}$ D = Dimension of the object

What is the length of a coastline?



cover the curve with "boxes" of length ε, and count how many you need

$$L = N(\varepsilon) \cdot \varepsilon$$

If we let $\varepsilon \to \frac{1}{2}\varepsilon$, we expect $N(\varepsilon) \to 2N(\frac{\varepsilon}{2})$ don't we?
But the smaller ruler measures caves and coves, so the length increases more than ×2.

$$N(\varepsilon) \sim \frac{L}{\varepsilon^D} \Rightarrow \ln N(\varepsilon) \sim \text{const} - D \ln \varepsilon$$

$$= \text{const} + D \ln(1/\varepsilon)$$

$$\boxed{\therefore D = \lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)}} \qquad \lim_{\varepsilon \to 0} \frac{\text{const}}{\ln(1/\varepsilon)} \to 0$$

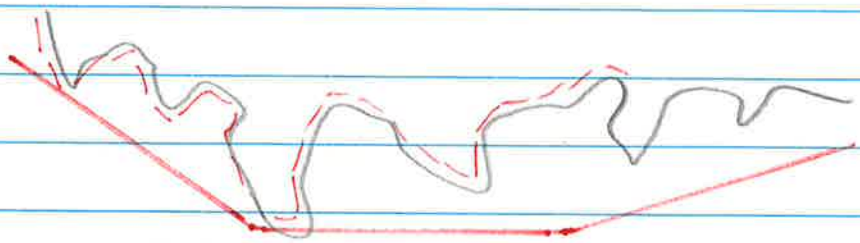
FRACTAL DIMENSION OF THE CURVE

$\Rightarrow \ln N(\varepsilon) = -D \ln \varepsilon$

$$\boxed{\therefore D = \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}}$$

BOX-COUNTING OR FRACTAL DIMENSION

Why coastline? They are wiggly!

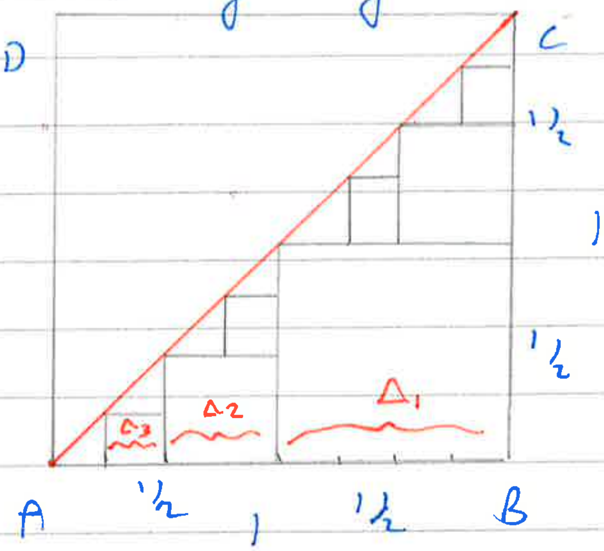


It turns out $D > 1$ i.e. $N(\varepsilon) \sim \frac{1}{\varepsilon^{1.25}}$

Non-Differentiable Functions

... or not all continuous curves have a length

Q. How long is the diagonal of a unit square?



$\vec{AC} = \vec{AB} + \vec{BC}$ has length 2

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$

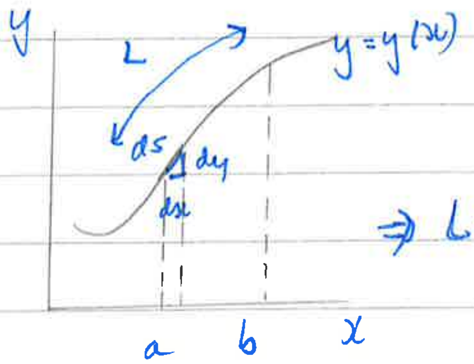
After N divisions: $\ldots 2^{N+1} \cdot \frac{1}{2^N}$

Let $\Delta_i = 2^{-i}$ $\qquad \lim_{N \to \infty} \sum_{i=1}^{N} \Delta_i = 2$

of Δs : 2^{i+1}

NB You can also use 1/3 and get a length of 3 for the diagonal.

But $|AC| = \sqrt{2}$! The curve is all corners!



$$ds^2 = dx^2 + dy^2$$

$$\Rightarrow L = \int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b dx \sqrt{1 + y'^2}$$

If y' doesn't exist, we cannot calculate the length!